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Quantum states which behave classically

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Abstract. We give a definition of quantum states which behave classically based on minimal number of physically reasonable requirements. We prove that an infinite unique class of states exists which satisfies this definition and we show that every state from this class may be generated in the unique way departing from some corresponding strictly quantum state. We discuss some implications of the obtained results.

1. Introduction

In all practical applications quantum mechanics works exceedingly well. Thus far no example has been found for which predictions of quantum mechanics are in conflict with experiment. In this respect quantum mechanics is one of the most reliable theories ever known. Yet quantum mechanics has some fundamental problems which are the subject of many controversies. One of them is the problem of the transition from quantum to classical mechanics. There is an enormous amount of literature devoted to this problem. It would take a lot of space just to mention all the authors who have dealt with this problem. We shall only point out that during the last few decades significant progress has been made in elucidating the problem, as described, for example, in the review of Omnes [1], but that a satisfactory solution has not yet been achieved. Consequently, any contribution which can clarify any aspect of the problem is of great value. The purpose of this paper is to try to clarify one important aspect of the relation between quantum and classical mechanics. It concerns the question of when we can say that a given quantum state behaves classically. In the current literature one can often find statements that some quantum states behave classically, but, to our knowledge, there is no explicit, precise and general criterion which would enable one to test whether a given quantum state behaves classically or not. In fact, such a criterion should, at the same time, give a precise meaning to the very term 'classically behaved quantum state'. We think that this is a necessary intermediate step toward the ultimate understanding of the relation between quantum and classical mechanics. In the next section we give one such criterion and find the unique infinite class of states which satisfy this criterion. In the last section we discuss the obtained results.

2. Theory

As quantum mechanics is inherently a statistical theory, a natural approach to investigate its classical limit is to compare it with classical statistical mechanics. The easiest way to do this is to describe a quantum state by its corresponding quantum phase-space distribution.

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In classical statistical mechanics the physical state is described by the non-negative phase-space distribution $\rho(q, p)$. The average value of any physical quantity $f(q, p)$, which is a function of coordinate and momentum, may be represented in the form

$$\langle f\rangle_{\mathrm{cl}} = \int f(q, p)\rho(q, p)\,\mathrm{d}p\,\mathrm{d}q.$$

In the standard formulation of quantum mechanics the average value for the same physical quantity, in the state described by the density matrix $\hat{\rho}$, is given by

$$\langle f\rangle_{\mathrm{q.m.}} = \mathrm{tr}\,\hat{\rho}\hat{f}$$

where $\hat{f}$ is the quantum-mechanical operator corresponding by the accepted rule of quantization to the classical function $f(q, p)$.

In phase-space formulations of quantum mechanics, average values may be represented in a structurally equivalent form to the classical one, with phase-space quasidistribution having the role of the classical probability distribution, after ascribing to the quantum operator $\hat{f}$ the function $\tilde{f}(q, p)$, which corresponds to the operator $\hat{f}$ in this phase-space formulation. However, except in Wigner phase-space formulation of quantum mechanics, the function $\tilde{f}(q, p)$ differs from the original function $f(q, p)$ from which the quantum operator $\hat{f}$ was obtained.

Obviously, the quantum state may be considered to behave classically in the sense of classical statistical mechanics if the following conditions are fulfilled.

(1) The quantum phase-space distribution describing this state is non-negative.

(2) The exact quantum-mechanical average values for all physical quantities in this state are, to the required accuracy, equal to the average values obtained using the formulae of classical statistical mechanics, where the considered quantum-phase distribution plays the role of the classical phase-space probability distribution and the physical quantity whose average value is calculated is represented by the classical phase-space function—the same one from which the quantum operator of this quantity was derived.

(3) The quantum phase-space distribution evolves in time to a given accuracy according to the classical Liouville equation.

By our definition the quantum state behaves classically if it fulfils these three conditions. The definition is justified by the fact that all classical states, in the sense of classical statistical mechanics, are described by non-negative phase distributions which evolve in time according to the Liouville equation and that all information about the state may be obtained by finding average values of various physical quantities.

Now we will give one example of such quantum-mechanical states and then prove that this is the unique class of such states.

For simplicity, we treat the one-dimensional case.

The first condition is, by definition, satisfied by every Husimi distribution [2], however, exact quantum-mechanical average values differ from average values obtained from such distributions in the classical way in the sense just explained [3]. When Wigner functions are used for the description of a state, quantum-mechanical average values are calculated by formulae which in their structure are identical with classical formulae in which the Wigner function plays the role of probability distribution. However, the Wigner function is not non-negative, and in the general case does not satisfy the classical Liouville equation [4, 5].

Let us consider now any quantum state whose Husimi distribution is $D(q, p)$. Transforming this function to $\lambda^2 D(\lambda q, \lambda p)$ where $\lambda < 1$, the new quantum state will be obtained, but the transformed function will again be the Husimi distribution [6]. Now, between the Wigner function and the Husimi function exists the relation [7]

$$W(q, p) = exp\left[-\frac{1}{4}\frac{\partial^2}{\partial q^2} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right] D(q, p). \tag{1}$$

Note that we have set $\hbar = M\omega = 1$.

If we take sufficiently small λ which depends of the required accuracy, and because every differentiation with respect to q and p introduces multiplication by the small parameter λ, we shall have, for λ-transformed states to the second order in λ

$$W(q, p) \approx \lambda^2 D(\lambda q, \lambda p). \tag{2}$$

Now, exact quantum-mechanical average values for any dynamical function $f(q, p)$ in a state described by $W(q, p)$ are given by [5]

$$\langle f \rangle = \int f(q, p) W(q, p)\, \mathrm{d}q\, \mathrm{d}p \tag{3}$$

and for λ-transformed states we have according to (2) and (3)

$$\langle f \rangle \approx \int f(q, p) \lambda^2 D(\lambda q, \lambda p)\, \mathrm{d}q\, \mathrm{d}p. \tag{4}$$

This formula represents the quantum-mechanical average value to the same accuracy to which (2) is valid. It has exactly the classical structure so that for λ-transformed states our first two conditions are fulfilled.

The time evolution of the Husimi distribution is described by the equation [8, 3]

$$\frac{\partial D(q, p)}{\partial t} = \frac{1}{M}\left[p + \frac{1}{2}\frac{\partial}{\partial p}\right]\frac{\partial D(q, p)}{\partial q} + \sum_{n}^{\infty}\sum_{m}^{\infty}\sum_{k=0}^{[m/2]} \frac{(\mathrm{i}\hbar)^{n-1}}{2^{n+m-1} n! k! (m-2k)!} \frac{\partial^{n+m} V(q)}{\partial q^{n+m}} \frac{\partial^n}{\partial p^n} \frac{\partial^{m-2k}}{\partial q^{m-2k}} D(q, p). \tag{5}$$

For λ-transformed states, neglecting the derivatives of distribution function of the second and higher orders, this equation reduces to the classical Liouville equation

$$\frac{\partial \lambda^2 D(\lambda q, \lambda p)}{\partial t} = -\frac{p}{M}\frac{\partial \lambda^2 D(\lambda q, \lambda p)}{\partial q} + \frac{\partial V(q)}{\partial q}\frac{\partial \lambda^2 D(\lambda q, \lambda p)}{\partial p}. \tag{6}$$

So, λ-transformed states also fulfil our third condition.

This means that both in accord with our definition and the general physical intuition and practice, these states belong to the class of quantum states which behave classically. In this way our definition has a non-empty domain.

Are there some different, non-λ-transformed states, which satisfy all the above three conditions?

We will prove now that all quantum-mechanical states which satisfy the above stated three conditions, i.e. which behave classically, may be represented as λ-transformed states.

A Husimi function which behaves classically must be slowly varying function such that its derivatives must be much smaller than the function itself because only in this case average values of any quantity $f(q, p)$ may be obtained with the required accuracy by using classical formulae.

Since every Husimi function may be represented in the form

$$f(q, p)\mathrm{e}^{-\beta p^2 - \gamma q^2} \tag{7}$$

both $f(q, p)$ and the exponential function must be slowly varying and this implies that β and γ must be small.

Now, as (7) is a Husimi distribution the expression

$$f\left(\mathrm{i}\frac{(p_1-p_2)}{2}, \frac{p_1+p_2}{2}\right)\exp\left[-\beta\frac{(p_1+p_2)^2}{4}-\gamma\frac{(p_1-p_2)^2}{4}-\frac{(p_1-p_2)^2}{4}\right] \tag{8}$$

must be positive definite [9]. In order that the function

$$f(\mu q, \mu p)\mathrm{e}^{-\beta\mu^2p^2-\gamma\mu^2q^2}$$

which is a Husimi distribution for $\mu = 1$ remains a Husimi distribution for $\mu > 1$ the following expression must be positive definite [9]

$$f\left(\mathrm{i}\frac{\mu}{2}(p_1-p_2), \frac{\mu}{2}(p_1+p_2)\right)\exp\left[-\beta\mu^2\frac{(p_1+p_2)^2}{4}-\gamma\mu^2\frac{(p_1-p_2)^2}{4}-\frac{(p_1-p_2)^2}{4}\right]. \tag{9}$$

Comparing this expression with (8) it is obvious that the first factor after transformation remains positive definite. The second factor, which is a Gaussian function, will be positive definite when [9]

$$1-\mu^2(\gamma+\beta) > 0$$

i.e

$$\mu^2 < \frac{1}{\gamma+\beta}. \tag{10}$$

As $\gamma+\beta$ is a small number so that $\frac{1}{\gamma+\beta} \gg 1$, the inequality (10) may be satisfied when μ is much greater than unity and this means that every Husimi function which behaves classically may be represented as λ-transformed state.

3. Conclusion and discussion

We have proved that λ-transformed states behave classically and that all states which behave classically in the sense of our definition, may be represented as λ-transformed states of states which are essentially non-classical. This last fact has an essential consequence. Namely, according to one of our earlier results [10], all λ-transformed states are unavoidably mixed states. This means that pure states can never behave classically in all respects. A pure state can never become a mixed state without interaction with its surroundings. From this it follows that a classically behaved state must necessarily have non-negligible interaction with its surrounding, at least in one part of its evolution. The analogous conclusion was reached by various authors in different ways [1].

It is important to underline, that λ-transformation of a Husimi distribution is not merely a formal mathematical transformation, but that this transformation may be interpreted as the Glauber most quiet phase-insensitive amplification of the initial state [10]. So, Glauber amplification process [11–13] may be considered as one possible concrete model for decoherence of a quantum state but of course not the unique model.

Although, for simplicity, we restricted our considerations to the one-dimensional case, the obtained results may be generalized to three space dimensions in a way analogous to that used in [14] for Wigner functions. Also, the results are valid for all physical systems which may be described by non-relativistic Hamiltonians which are functions of coordinate and momentum of any form.

Our approach is limited to the non-relativistic physical systems and so is not applicable to spin degrees of freedom, because spin does not have a classical analogon so that our condition (2) cannot be applied. However, when the state is such that the spin dependence factorizes, as is the case in non-relativistic situations, all our arguments remain valid for physical quantities which have a classical analogon.

Let us note that although our way of presentation was such that the Weyl quantization was tacitly assumed, equation (3), the results are not sensitive to the ordering ambiguity. Namely, when the state is described by the Husimi function, in the expressions for exact quantum-mechanical average values, the same classical term appears in every ordering [3] while the other terms, which are different for various orderings, always contain some powers of Planck constant as multiplicative factors. Due to this for small λ—i.e. in the classical limit—all of them, in every ordering, become negligibly small compared with the classical one, and only the classical term which is the same in all orderings survives in this limit.

It should be noted, as may be easily verified, that the portion of phase space over which λ-transformed states extend is approximately $\frac{1}{\lambda^2}$ times larger than that for the corresponding initial states. Assuming now that λ is such that these states may be considered to behave classically, an interesting question arises concerning how narrow they can be or, equivalently, how close to unity λ can be. A general quantitative answer to this question cannot be given. Evidently this depends on the overall accuracy required for the fulfilment of our condition (2) and various physical situations, depending on the problems which may be considered, may require different accuracies.

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